Dynare Summer School 2021

Introduction to Dynare - correction of exercise session 1
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For these exercises, you will use the examples contained in `Session1.zip`.

Exercise 1: Stochastic model For this exercise, we use the small scale New Keynesian model from Sungbae An and Frank Schorfheide (2006) *Bayesian Analysis of DSGE models.* Federal Reseerve Bank of Philadelphia, WP No. 06-5, (available in `papers/an_schorfheide_2006.pdf`) and Edward Herbst and Frank Schorfheide (2016) *Bayesian Estimation of DSGE models.* Princeton University Press.
Look at these different representations of the model (refer to the original paper for the meaning of the variables and parameters):

1. Model A, the original nonlinear model contains the following equations:

$$
\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right] && (1) \\
1 &= \phi(\pi_t - \pi) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\pi}{2\nu} \right] \\
&\quad - \phi\beta \mathbb{E}_t \left[\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t} \right)^{-\tau} \frac{Y_{t+1}/A_{t+1}}{Y_t/A_t} (\pi_{t+1} - \pi)\pi_{t+1} \right] && (2) \\
&\quad + \frac{1}{\nu} \left[1 - \left(\frac{C_t}{A_t} \right)^{\tau} \right] \\
Y_t &= C_t + G_t + AC_t && (3) \\
AC_t &= \frac{\phi}{2} (\pi_t - \pi)^2 Y_t && (4) \\
G_t &= \frac{g_t - 1}{g_t} Y_t && (5) \\
R_t &= R_t^{\star 1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}} && (6) \\
R_t^{\star} &= r\pi^{\star} \left(\frac{\pi_t}{\pi^{\star}} \right)^{\psi_1} \left(\frac{Y_t}{Y_t^{\star}} \right)^{\psi_2} && (7) \\
\ln A_t &= \ln \gamma + \ln A_{t-1} + \ln z_t && (8) \\
\ln z_t &= \rho_z \ln z_{t-1} + \epsilon_{z,t} && (9) \\
\ln g_t &= (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t} && (10) \\
Y_t^{\star} &= (1 - \nu)^{\frac{1}{\tau}} A_t g_t && (11)
\end{aligned}
$$

2. Model B is the stationary version of the model. It is obtained by removing the productivity trend from consumption, $c_t = C_t/A_t$, outptut, $y_t = Y_t/A_t$, and, natural output, $y_t^\star = Y_t^\star/A_t$. In addition, we substitute out G_t and AC_t:

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma z_{t+1}} \frac{R_t}{\pi_{t+1}} \right] \tag{12}$$

$$\begin{aligned} 1 = {} & \phi(\pi_t - \pi) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\pi}{2\nu} \right] \\ & - \phi\beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \pi) \pi_{t+1} \right] \\ & + \frac{1}{\nu} \left(1 - c_t^\tau \right) \end{aligned} \tag{13}$$

$$y_t = c_t + \frac{g_t - 1}{g_t} y_t + \frac{\phi}{2} (\pi_t - \pi)^2 y_t \tag{14}$$

$$R_t = R_t^{\star 1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}} \tag{15}$$

$$R_t^\star = r\pi^\star \left(\frac{\pi_t}{\pi^\star} \right)^{\psi_1} \left(\frac{y_t}{y_t^\star} \right)^{\psi_2} \tag{16}$$

$$\ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t} \tag{17}$$

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t} \tag{18}$$

$$y_t^\star = (1 - \nu)^{\frac{1}{\tau}} g_t \tag{19}$$

3. Steady state of the stationary model is

$$\pi = \pi^\star \tag{20}$$

$$r = \frac{\gamma}{\beta} \tag{21}$$

$$R = r\pi^\star \tag{22}$$

$$c = (1 - \nu)^{\frac{1}{\tau}} \tag{23}$$

$$y^\star = gc \tag{24}$$

$$y = y^\star \tag{25}$$

4. Model C is obtained by using variables defined as ratio to the steady state value, $\hat{x}_t = \ln\left(\frac{x_t}{x}\right)$:

$$1 = \mathbb{E}_t\left[e^{-\tau\hat{c}_{t+1}+\tau\hat{c}_t+\hat{R}_t-\hat{z}_{t+1}-\hat{\pi}_{t+1}}\right] \tag{26}$$

$$\begin{aligned}0 = &\left(e^{\hat{\pi}_t}-1\right)\left[\left(1-\frac{1}{2\nu}\right)e^{\hat{\pi}_t}+\frac{1}{2\nu}\right]\\ &-\beta\mathbb{E}_t\left[\left(e^{\hat{\pi}_{t+1}}-1\right)e^{-\tau\hat{c}_{t+}+\tau\hat{c}_t+\hat{y}_{t+1}-\hat{y}_t+\hat{\pi}_{t+1}}\right]\\ &+\frac{1-\nu}{\nu\phi\pi^2}\left(1-e^{\tau\hat{c}_t}\right)\end{aligned} \tag{27}$$

$$e^{\hat{c}_t-\hat{y}_t} = e^{-\hat{g}_t}-\frac{\phi\pi^2 g}{2}\left(e^{\hat{\pi}_t}-1\right)^2 \tag{28}$$

$$\begin{aligned}\hat{R}_t = &\rho_R\hat{R}_{t-1}+(1-\rho_R)\psi_1\hat{\pi}_t\\ &+(1-\rho_R)\psi_2(\hat{y}_t-\hat{g}_t)+\epsilon_{R,t}\end{aligned} \tag{29}$$

$$\hat{g}_t = \rho_g\hat{g}_{t-1}+\epsilon_{g,t} \tag{30}$$

$$\hat{z}_t = \rho_z\hat{z}_{t-1}+\epsilon_{z,t} \tag{31}$$

5. Finally, model D is the log-linearized version of the original model

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}]-\frac{1}{\tau}\left(\hat{R}_t-\mathbb{E}_t[\hat{\pi}_{t+1}]-\mathbb{E}_t\left[\hat{z}_{t+1}\right]\right) \tag{32}$$

$$+\hat{g}_t-\mathbb{E}_t\left[\hat{g}_{t+1}\right] \tag{33}$$

$$\pi_t = \beta\mathbb{E}_t[\hat{\pi}_{t+1}]+\kappa(\hat{y}_t-\hat{g}_t) \tag{34}$$

$$\hat{R}_t = \rho_R\hat{R}_{t-1}+(1-\rho_R)(\psi_1\hat{\pi}_t+\psi_2(\hat{y}_t-\hat{g}_t))+\epsilon_{R,t} \tag{35}$$

$$\hat{g}_t = \rho_g\hat{g}_{t-1}+\epsilon_{g,t} \tag{36}$$

$$\hat{z}_t = \rho_z z_{t-1}+\epsilon_{z,t} \tag{37}$$

with

$$\kappa = \tau\frac{1-\nu}{\nu\pi^2\phi}$$

Questions:

1. Would model A and model B generate the same trajectories for the endogenous variables?

   - No, in model A, Y_t, $Y_t^\star$, C_t, and G_t would be trending, while in model B, y_t, $y_t^\star$ and c_t would be stationary.

2. Would model B and model C generate the same trajectories for the endogenous variables?

   - No, variables in model C behave as the log of variables in model B. In addition, in model B, variables have distinct steady state values depending on the parameters of the model, while in model C, the steady state of all variables is zero.

3. Check the transformation of the first equation from its original form, equation (1), to its expression in terms of stationary variables, equation (12).

   (a) In equation (1), replace $C_{t+1}/A_{t+1} = c_{t+1}$ and $C_t/A_t = c_t$.

   (b) From equation (5)

   $$A_{t+1} = \gamma A_t z_{t+1}$$
   $$\frac{A_t}{A_{t+1}} = \frac{1}{\gamma z_{t+1}}$$

   (c) R_t and π_{t+1} don't change as they are not affected by the productivity trend.

4. Model B is available in `models/as/as1.mod`. Fill in the missing equation and run `as1.mod` in Dynare.

   - See `corrections/models/as/as1_corr.mod`

5. Verify equation (22)

   (a) Consider equation (12). From equation (17), the steady state value of z_t is equal to 1.

   (b) At the steady state, $\frac{c_{t+1}}{c_t} = 1$ and equation (12) reduces to

   $$1 = \frac{\beta R}{\gamma \pi^\star}.$$

   Therefore,

   $$R = \frac{\gamma \pi^\star}{\beta}$$

6. Check the transformation of equation (12) to equation (26)

   (a) Introduce $\hat{c}_t$, $\hat{R}_t$, $\hat{\pi}_t$ and $\hat{z}_t$:

$$
\begin{aligned}
1 &= \beta \mathbb{E}_t \left[\left(\frac{c e^{\hat{c}_{t+1}}}{c e^{\hat{c}_t}} \right)^{-\tau} \frac{1}{\gamma e^{\hat{z}_t}} \frac{R e^{\hat{R}_t}}{\pi e^{\hat{\pi}_{t+1}}} \right] \\
&= \beta \mathbb{E}_t \left[e^{\tau \hat{c}_{t+1} - \tau \hat{c}_t - \hat{z}_t + \hat{R}_t - \hat{\pi}_{t+1}} \left(\frac{R}{\gamma \pi} \right) \right] \\
&= \mathbb{E}_t \left[e^{\tau \hat{c}_{t+1} - \tau \hat{c}_t - \hat{z}_t + \hat{R}_t - \hat{\pi}_{t+1}} \right]
\end{aligned}
$$

   as $R = \frac{\gamma \pi}{\beta}$

7. Model C is implemented in `models/as/as2.mod` and a first order approximation is computed. Run it and compare the mean of the variables between `as1.mod` and `as2.mod`. Where is the difference coming from?

   - In model C, the variables are expressed as relative gap from the steady state. In a linearized model appproximation, the mean gap is zero.

8. Model D is implemented in `models/as/as3.mod`. Run it and compare the results with `as2.mod`?

   - The results are identical. In model C, Dynare computes automically the first order approximation that is computed by hand in model D.

Exercise 2: Higher order approximations

1. In `models/jermann98/jermann98.mod`: change the value of the parameters for consumption habits and investment adjustment cost and observe the effect on the risk premium. Try second and third order approximation.

   - In `corrections/jermann98/jermann98a.mod`, rigidities are increased by increasing `chihab` and `xi` and a third order approximation is computed. Increased rigidities result in a smaller variance of consumption and a smaller mean risk premium.

Exercise 3: Linearization

1. Compute the first order Taylor expansion of equation (26)

    - The expansion is computed around the steady state. At the steady state, $\hat{c}_t$, $\hat{R}_t$, $\hat{z}_t$, $\hat{\pi}_t$ equal zero and therefore

    $$e^{-\tau\hat{c}_{t+1}+\tau\hat{c}_t+\hat{R}_t-\hat{z}_{t+1}-\hat{\pi}_{t+1}} = 1$$

    and

    $$\begin{aligned}
    \frac{\partial e^{-\tau\hat{c}_{t+1}+\tau\hat{c}_t+\hat{R}_t-\hat{z}_{t+1}-\hat{\pi}_{t+1}}}{\partial \hat{c}_{t+1}} &= -\tau \\
    \frac{\partial e^{-\tau\hat{c}_{t+1}+\tau\hat{c}_t+\hat{R}_t-\hat{z}_{t+1}-\hat{\pi}_{t+1}}}{\partial \hat{c}_t} &= \tau \\
    \frac{\partial e^{-\tau\hat{c}_{t+1}+\tau\hat{c}_t+\hat{R}_t-\hat{z}_{t+1}-\hat{\pi}_{t+1}}}{\partial \hat{R}_t} &= 1 \\
    \frac{\partial e^{-\tau\hat{c}_{t+1}+\tau\hat{c}_t+\hat{R}_t-\hat{z}_{t+1}-\hat{\pi}_{t+1}}}{\partial \hat{z}_{t+1}} &= -1 \\
    \frac{\partial e^{-\tau\hat{c}_{t+1}+\tau\hat{c}_t+\hat{R}_t-\hat{z}_{t+1}-\hat{\pi}_{t+1}}}{\partial \hat{\pi}_{t+1}} &= -1
    \end{aligned}$$

    - The first order Taylor expansion of equation (26) is

    $$0 = -\tau\mathbb{E}_t\left[\hat{c}_{t+1}\right] + \tau\hat{c}_t + \hat{R}_t - \mathbb{E}_t\left[\hat{z}_{t+1}\right] - \mathbb{E}_t\left[\hat{\pi}_{t+1}\right] \tag{38}$$

2. Compute the first order Taylor expansion of equation (28)

    - Again,

    $$\begin{aligned}
    e^{\hat{c}_t-\hat{y}_t} &= 1 \\
    e^{-\hat{g}_t} &= 1 \\
    e^{\hat{\pi}_t} &= 1
    \end{aligned}$$

    - The first order Taylor expansion of equation (28) is

    $$\hat{c}_t - \hat{y}_t = \hat{g}_t \tag{39}$$

3. Combine the first order Taylor expansion of equations (26) and (28) to obtain equation (32)

- From equation (39):

$$\hat{c}_t = \hat{y}_t - \hat{g}_t$$

- Replacing $\hat{c}_t$ in equation (38)

$$\begin{aligned}
0 &= -\tau \left(\mathbb{E}_t \left[\hat{y}_{t+1} \right] - \mathbb{E}_t \left[\hat{g}_{t+1} \right] \right) + \tau (\hat{y}_t - \hat{g}_t) + \hat{R}_t - \mathbb{E}_t \left[\hat{z}_{t+1} \right] - \mathbb{E}_t \left[\hat{\pi}_{t+1} \right] \\
\hat{y}_t &= \mathbb{E}_t [\hat{y}_{t+1}] - \frac{1}{\tau} \left(\hat{R}_t - \mathbb{E}_t [\hat{\pi}_{t+1}] - \mathbb{E}_t [\hat{z}_{t+1}] \right) \\
&\quad + \hat{g}_t - \mathbb{E}_t \left[\hat{g}_{t+1} \right]
\end{aligned}$$

4. Again, Dynare can compute any linearization for you.