First order approximation

For

$$\begin{aligned} &E_t\left\{F(y_{t-1},u_t,\epsilon_{t+1},\sigma)\right\}\\ &= E_t\left\{f(g(g(y_{t-1},u_t,\sigma),\sigma\epsilon_{t+1},\sigma),g(y_{t-1},u_t,\sigma),y_{t-1},u_t)\right\}\\ &=0\end{aligned}$$

we develop the Taylor expansion at $y_t = y_{t-1} = y_{t+1} = \bar{y}$ and $u_t = \epsilon_t = 0$:

$$\begin{aligned} E_t\left\{F^{(1)}(y_{t-1},u_t,\epsilon_{t+1},\sigma)\right\} = \\ & E_t\Big\{f(\bar{y},\bar{y},\bar{y},0)+f_{y_+}\left(g_y\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+g_u\sigma\epsilon'+g_\sigma\sigma\right)\\ & +f_{y_0}\left(g_y\hat{y}+g_u u+g_\sigma\sigma\right)+f_{y_-}\hat{y}+f_u u\Big\}\\ = & \quad 0\end{aligned}$$

with $\hat{y} = y_{t-1}-\bar{y}$, $u = u_t$, $\epsilon' = \epsilon_{t+1}$, $f_{y_+} = \frac{\partial f}{\partial y_{t+1}}$, $f_{y_0} = \frac{\partial f}{\partial y_t}$, $f_{y_-} = \frac{\partial f}{\partial y_{t-1}}$, $f_u = \frac{\partial f}{\partial u_t}$, $g_y = \frac{\partial g}{\partial y_{t-1}}$, $g_u = \frac{\partial g}{\partial u_t}$, $g_\sigma = \frac{\partial g}{\partial \sigma}$.

Certainty equivalence

$$
\begin{aligned}
&E_t\Big\{f(\bar{y},\bar{y},\bar{y},0) + f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + g_u\sigma\epsilon' + g_\sigma\sigma\right)\\
&\qquad + f_{y_0}\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + f_{y_-}\hat{y} + f_u u\Big\}\\
&= f(\bar{y},\bar{y},\bar{y},0) + f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + g_u\sigma E_t\epsilon' + g_\sigma\sigma\right)\\
&\qquad + f_{y_0}\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + f_{y_-}\hat{y} + f_u u\\
&= f(\bar{y},\bar{y},\bar{y},0) + f_{y_+}\left(g_y\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + g_u\sigma + g_\sigma\sigma\right)\\
&\qquad + f_{y_0}\left(g_y\hat{y} + g_u u + g_\sigma\sigma\right) + f_{y_-}\hat{y} + f_u u\\
&= 0
\end{aligned}
$$

Taking the expectation

$$
\begin{aligned}
E_t \left\{ F^{(1)}(y_{t-1}, u_t, \epsilon_{t+1}, \sigma) \right\} = &\\
& f(\bar{y}, \bar{y}, \bar{y}, 0) + f_{y_+} \left(g_y \left(g_y \hat{y} + g_u u + g_\sigma \sigma \right) + g_\sigma \sigma \right) \\
& + f_{y_0} \left(g_y \hat{y} + g_u u + g_\sigma \sigma \right) + f_{y_-} \hat{y} + f_u u \Big\} \\
= \quad & \left(f_{y_+} g_y g_y + f_{y_0} g_y + f_{y_-} \right) \hat{y} + \left(f_{y_+} g_y g_u + f_{y_0} g_u + f_u \right) u \\
& + \left(f_{y_+} \left(g_y g_\sigma + g_\sigma \right) + f_{y_0} g_\sigma \right) \sigma \\
= \quad & 0
\end{aligned}
$$

Recovering g_y

$$\left(f_{y_+} g_y g_y + f_{y_0} g_y + f_{y_-}\right) \hat{y} = 0$$

- This is matrix quadratic equation
- There is a multiplicity of solutions for g_y but only some of them are stable (going back to the steady state after a shock)
- In general, we prefer models where there exist a unique stable solution and Dynare works only with those.
- If your model doesn't have a any stable solution or a multiplicity of stable solutions, Dynare issues an error.
- Dynare uses two different algorithms to solve matrix quadratic equations:
  - Generalized Schur Decomposition (default)
  - Cyclic reduction
- The description of these algorithms are in Appendix

Blanchard and Kahn condition

- The Blanchard and Kahn condition for the existence of a unique stable solution in a Linear Rational Expectation model: *there must be as many forward-looking variables in the model as there are eigenvalues larger than one in modulus*.

Recovering g_u

$$f_{y_+} g_y g_u + f_{y_0} g_u + f_u = 0$$

This is a simple linear system:

$$g_u = -\left(f_{y_+} g_y + f_{y_0}\right)^{-1} f_u$$

Recovering g_σ

$$f_{y_+} g_y g_\sigma + f_{y_0} g_\sigma = 0$$

This is an homogeneous linear system:

$$g_\sigma = 0$$

Yet another manifestation of the certainty equivalence property of first order approximation.

First order approximated decision function

$$y_t = \bar{y} + g_y \hat{y} + g_u u$$

$$\begin{aligned} E\{y_t\} &= \bar{y} \\ \Sigma_y &= g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \end{aligned}$$

The variance is solved for with an algorithm for discrete time Lyapunov equations.