A matrix quadratic equation

Consider the equation

$$\left(f_{y_+} g_y g_y + f_{y_0} g_y + f_{y_-}\right) \hat{y} = 0$$

Two approaches

- The generalized Schur decomposition
- The cyclic reduction approach

Generalized Schur decomposition approach

Consider the structural state space representation:

$$\begin{bmatrix} 0 & f_{y_+} \\ I & 0 \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} = \begin{bmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y}$$

or

$$\begin{bmatrix} 0 & f_{y_+} \\ I & 0 \end{bmatrix} \begin{bmatrix} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{bmatrix} = \begin{bmatrix} -f_{y_-} & -f_{y_0} \\ 0 & I \end{bmatrix} \begin{bmatrix} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{bmatrix}$$

Structural state space representation

$$D x_{t+1} = E x_t$$

with

$$x_{t+1} = \begin{bmatrix} y_t - \bar{y} \\ y_{t+1} - \bar{y} \end{bmatrix} \qquad x_t = \begin{bmatrix} y_{t-1} - \bar{y} \\ y_t - \bar{y} \end{bmatrix}$$

- There are multiple solutions but we want a unique stable one.
- Need to discuss eigenvalues of this linear system.
- Problem when D is singular.

Real generalized Schur decomposition

Taking the real generalized Schur decomposition of the pencil $< E, D >$:

$$
\begin{aligned}
D &= QTZ \\
E &= QSZ
\end{aligned}
$$

with T, upper triangular, S quasi-upper triangular, $Q'Q = I$ and $Z'Z = I$.

Generalized eigenvalues

λ_i solves

$$\lambda_i D x_i = E x_i$$

For diagonal blocks on S of dimension 1 x 1:

- $T_{ii} \neq 0$: $\lambda_i = \frac{S_{ii}}{T_{ii}}$
- $T_{ii} = 0$, $S_{ii} > 0$: $\lambda_i = +\infty$
- $T_{ii} = 0$, $S_{ii} < 0$: $\lambda_i = -\infty$
- $T_{ii} = 0$, $S_{ii} = 0$: $\lambda_i \in \mathcal{C}$

A pair of complex eigenvalues

When a diagonal block of matrix S is a 2x2 matrix of the form
$\begin{bmatrix} S_{ii} & S_{i,i+1} \\ S_{i+1,i} & S_{i+1,i+1} \end{bmatrix}$,

- the corresponding block of matrix T is a diagonal matrix,
- $(S_{i,i}T_{i+1,i+1} + S_{i+1,i+1}T_{i,i})^2 < -4S_{i+1,i}S_{i+1,i}T_{i,i}T_{i+1,i+1}$,
- there is a pair of conjugate eigenvalues

$$\lambda_i, \lambda_{i+1} =$$

$$\frac{S_{ii}T_{i+1,i+1} + S_{i+1,i+1}T_{i,i} \pm \sqrt{(S_{i,i}T_{i+1,i+1} - S_{i+1,i+1}T_{i,i})^2 + 4S_{i+1,i}S_{i+1,i}T_{i,i}T_{i+1,i+1}}}{2T_{i,i}T_{i+1,i+1}}$$

Applying the decomposition

$$
\begin{aligned}
D \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} &= E \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y} \\
\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} g_y \hat{y} & \\
&= \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I \\ g_y \end{bmatrix} \hat{y}
\end{aligned}
$$

Selecting the stable trajectory

To exclude explosive trajectories, one imposes

$$Z_{21} + Z_{22} g_y = 0$$

$$g_y = -Z_{22}^{-1} Z_{21}$$

A unique stable trajectory exists if Z_{22} is non-singular: there are as many roots larger than one in modulus as there are forward–looking variables in the model (Blanchard and Kahn condition) and the rank condition is satisfied.

An alternative algorithm: Cyclic reduction

- Solving

$$A_0 + A_1 X + A_2 X^2$$

- Iterate

$$\begin{aligned}
A_0^{(k+1)} &= -A_0^{(k)}(A_1^{(k)})^{-1}A_0^{(k)}, \\
A_1^{(k+1)} &= A_1^{(k)} - A_0^{(k)}(A_1^{(k)})^{-1}A_2^{(k)} - A_2^{(k)}(A_1^{(k)})^{-1}A_0^{(k)}, \\
A_2^{(k+1)} &= -A_2^{(k)}(A_1^{(k)})^{-1}A_2^{(k)}, \\
\widehat{A}_1^{(k+1)} &= \widehat{A}_1^{(k)} - A_2^{(k)}(A_1^{(k)})^{-1}A_0^{(k)}.
\end{aligned}$$

  for $k = 1, \dots$ with $A_0^{(1)} = A_0$, $A_1^{(1)} = A_1$, $A_2^{(1)} = A_2$, $\widehat{A}_1^{(1)} = A_1$ and until $||A_0^{(k)}||_\infty < \epsilon$ and $||A_2^{(k)}||_\infty < \epsilon$.

- Then

$$X \approx -(\widehat{A}_1^{(k+1)})^{-1}A_0$$