A simple RBC model

Consider the following model of an economy.

- Representative agent preferences

$$U = \sum_{t=1}^{\infty} \beta^{t-1} E_t \left[\ln(C_t) - \frac{L_t^{1+\gamma}}{1+\gamma} \right]$$

  The household supplies labor and rents capital to the corporate sector.

  - L_t is labor services
  - β is the discount rate
  - $\gamma \in (0, \infty)$ is a labor supply parameter.
  - C_t is consumption,

RBC Model (continued)

- The household faces the sequence of budget constraints

$$K_t = (1-\delta) K_{t-1} + w_t L_t + r_t K_{t-1} - C_t,$$

  where
  - K_t is capital at the end of period
  - $\delta \in (0,1)$ is the rate of depreciation
  - w_t is the real wage,
  - r_t is the real rental rate

- The production function is given by the expression

$$Y_t = A_t K_{t-1}^{\alpha} \left((1+g)^t L_t\right)^{1-\alpha}$$

  where $g \in (0,\infty)$ is the growth rate of labor embodied technological change and α is a parameter.

RBC Model (continued)

- A_t is a technology shock that follows the process

$$A_t = A_{t-1}^{\lambda} \exp\left(e_t\right),$$

where e_t is an i.i.d. zero mean normally distributed error with standard deviation σ and $\lambda \in (0, 1)$ is a parameter.

The household problem

Lagrangian

$$L = \max_{C_t, L_t, K_t} \sum_{t=1}^{\infty} \beta^{t-1} E_t \Big[\ln (C_t) - \frac{L_t^{1+\gamma}}{1+\gamma}$$
$$- \mu_t \left(K_t - K_{t-1} (1-\delta) - w_t L_t - r_t K_{t-1} + C_t \right) \Big]$$

First order conditions

$$\frac{\partial L}{\partial C_t} = \beta^{t-1} \left(\frac{1}{C_t} - \mu_t \right) = 0$$
$$\frac{\partial L}{\partial L_t} = \beta^{t-1} \left(L_t^{\gamma} - \mu_t w_t \right) = 0$$
$$\frac{\partial L}{\partial K_t} = -\beta^{t-1} \mu_t + \beta^t E_t \left(\mu_{t+1} (1 - \delta + r_{t+1}) \right) = 0$$

First order conditions

Eliminating the Lagrange multiplier, one obtains

$$L_t^\gamma = \frac{w_t}{C_t}$$
$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} (r_{t+1} + 1 - \delta) \right)$$

The firm problem

$$\max_{L_t,K_{t-1}} A_t K_{t-1}^{\alpha} \left((1+g)^t L_t\right)^{1-\alpha} - r_t K_{t-1} - w_t L_t$$

First order conditions:

$$r_t = \alpha A_t K_{t-1}^{\alpha-1} \left((1+g)^t L_t\right)^{1-\alpha}$$
$$w_t = (1-\alpha) A_t K_{t-1}^{\alpha} \left((1+g)^t\right)^{1-\alpha} L_t^{-\alpha}$$

Goods market equilibrium

$$K_t + C_t = A_t K_{t-1}^{\alpha} \left((1+g)^t L_t \right)^{1-\alpha} + (1-\delta) K_{t-1}$$

Dynamic Equilibrium

$$
\begin{aligned}
\frac{1}{C_t} &= \beta E_t \left(\frac{1}{C_{t+1}} (r_{t+1} + 1 - \delta) \right) \\
L_t^{\gamma} &= \frac{w_t}{C_t} \\
r_t &= \alpha A_t K_{t-1}^{\alpha-1} \left((1+g)^t L_t \right)^{1-\alpha} \\
w_t &= (1-\alpha) A_t K_{t-1}^{\alpha} \left((1+g)^t \right)^{1-\alpha} L_t^{-\alpha} \\
K_t + C_t &= A_t K_{t-1}^{\alpha} \left((1+g)^t L_t \right)^{1-\alpha} + (1-\delta) K_{t-1}
\end{aligned}
$$

Existence of a balanced growth path

There must exist a growth rates g_c and g_k so that

$$(1+g_k)^t K_1 + (1+g_c)^t C_1 = \\ A\left(\frac{(1+g_k)^t}{1+g_k}K_0\right)^\alpha \left((1+g)^t L_1\right)^{1-\alpha} + (1-\delta)\frac{(1+g_k)^t}{1+g_K}K_0$$

for $t=1$. So,

$$g_c = g_k = g$$

and

$$K_1 + C_1 = A\left(\frac{1}{1+g}K_0\right)^\alpha L_1^{1-\alpha} + (1-\delta)\frac{1}{1+g}K_0$$

Stationarized model

Let's define

$$\widehat{C}_t = C_t/(1+g)^t$$
$$\widehat{K}_t = K_t/(1+g)^t$$
$$\widehat{w}_t = w_t/(1+g)^t$$

Stationarized model (continued)

$$\frac{1}{\widehat{C}_t(1+g)^t} = \beta E_t\left(\frac{1}{\widehat{C}_{t+1}(1+g)(1+g)^t}(r_{t+1}+1-\delta)\right)$$

$$L_t^\gamma = \frac{\widehat{w}_t(1+g)^t}{\widehat{C}_t(1+g)^t}$$

$$r_t = \alpha A_t\left(\widehat{K}_{t-1}\frac{(1+g)^t}{1+g}\right)^{\alpha-1}\left((1+g)^t L_t\right)^{1-\alpha}$$

$$\widehat{w}_t(1+g)^t = (1-\alpha)A_t\left(\widehat{K}_{t-1}\frac{(1+g)^t}{1+g}\right)^{\alpha}\left((1+g)^t\right)^{1-\alpha}L_t^{-\alpha}$$

$$\left(\widehat{K}_t+\widehat{C}_t\right)(1+g)^t = A_t\left(\widehat{K}_{t-1}\frac{(1+g)^t}{1+g}\right)^{\alpha}\left((1+g)^t L_t\right)^{1-\alpha} + (1-\delta)\widehat{K}_{t-1}\frac{(1+g)^t}{1+g}$$

Stationarized model (continued)

$$\frac{1}{\widehat{C}_t} = \beta E_t \left(\frac{1}{\widehat{C}_{t+1}(1+g)} (r_{t+1} + 1 - \delta) \right)$$

$$L_t^{\gamma} = \frac{\widehat{w}_t}{\widehat{C}_t}$$

$$r_t = \alpha A_t \left(\frac{\widehat{K}_{t-1}}{1+g} \right)^{\alpha-1} L_t^{1-\alpha}$$

$$\widehat{w}_t = (1-\alpha) A_t \left(\frac{\widehat{K}_{t-1}}{1+g} \right)^{\alpha} L_t^{-\alpha}$$

$$\widehat{K}_t + \widehat{C}_t = A_t \left(\frac{\widehat{K}_{t-1}}{1+g} \right)^{\alpha} L_t^{1-\alpha} + (1-\delta) \frac{\widehat{K}_{t-1}}{1+g}$$

Steady state

$$\frac{1}{\widehat{C}} = \beta\left(\frac{1}{\widehat{C}(1+g)}(r+1-\delta)\right)$$

$$L^\gamma = \frac{\widehat{w}}{\widehat{C}}$$

$$r = \alpha A\left(\frac{\widehat{K}}{1+g}\right)^{\alpha-1} L^{1-\alpha}$$

$$\widehat{w} = (1-\alpha)A\left(\frac{\widehat{K}}{1+g}\right)^{\alpha} L^{-\alpha}$$

$$\widehat{K} + \widehat{C} = A\left(\frac{\widehat{K}}{1+g}\right)^{\alpha} L^{1-\alpha} + (1-\delta)\frac{\widehat{K}}{1+g}$$

$$\ln A_t = \lambda \ln A + 0$$

Stready state: recursive computation

When $\gamma = 0$,

$$
\begin{aligned}
A &= 1 \\
r &= \frac{1+g}{\beta} + \delta - 1 \\
\widehat{KL} &= (1+g)\left(\frac{r}{\alpha A}\right)^{\frac{1}{\alpha-1}} \\
\widehat{w} &= (1-\alpha)A\left(\frac{\widehat{KL}}{1+g}\right)^{\alpha} \\
\widehat{C} &= \widehat{w} \\
L &= \frac{\widehat{C}}{A\widehat{KL}^{\alpha}(1+g)^{1-\alpha} - \delta\widehat{KL}} \\
\widehat{K} &= \widehat{KL} \cdot L
\end{aligned}
$$