Second and third order approximation of the model

- Second and third order approximation of the solution function are obtained from second, respectively third, order approximation of the model.
- It requires only the solution of (tricky) linear problems.
- The stochastic scale of the model, σ, appears in the solution and breaks certainty equivalence

Second and third order decision functions

- Second order

$$y_t = \bar{y} + \frac{1}{2}g_{\sigma\sigma}\sigma^2 + g_y\hat{y} + g_u u + \frac{1}{2}\left(g_{yy}(\hat{y}\otimes\hat{y}) + g_{uu}(u\otimes u)\right) + g_{yu}(\hat{y}\otimes u)$$

- Third order

$$y_t = \bar{y} + \frac{1}{2}g_{\sigma\sigma}\sigma^2 + \frac{1}{6}g_{\sigma\sigma\sigma}\sigma^3 + \frac{1}{2}g_{y\sigma\sigma}\hat{y}\sigma^2 + \frac{1}{2}g_{u\sigma\sigma}u\sigma^2 + g_y\hat{y} + g_u u + \frac{1}{2}\left(g_{yy}(\hat{y}\otimes\hat{y}) + g_{uu}(u\otimes u)\right) + g_{yu}(\hat{y}\otimes u) + \frac{1}{6}\left(g_{yyy}(\hat{y}\otimes\hat{y}\otimes\hat{y}) + g_{uuu}(u\otimes u\otimes u)\right) + \frac{1}{2}\left(g_{yyu}(\hat{y}\otimes\hat{y}\otimes u) + g_{yuu}(\hat{y}\otimes\hat{y}\otimes u)\right)$$

We can fix $\sigma = 1$.

Second order accurate moments

$$\begin{aligned}\Sigma_y &= g_y \Sigma_y g_y' + \sigma^2 g_u \Sigma_\epsilon g_u' \\ E\left\{y_t\right\} &= \bar{y} + (I - g_y)^{-1}\left(0.5\left(g_{\sigma\sigma} + g_{yy}\vec{\Sigma}_y + g_{uu}\vec{\Sigma}_\epsilon\right)\right)\end{aligned}$$

Further issues

- Impulse response functions depend of state at time of shocks and history of future shocks.
- For large shocks second order approximation simulation may explode
  - pruning algorithm (Sims)
  - truncate normal distribution (Judd)